

Location, identification, and representability of monotone operators in locally convex spaces

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Abstract

In this paper we study, in the relaxed context of locally convex spaces, intrinsic properties of monotone operators needed for the sum conjecture for maximal monotone operators to hold under classical interiority-type domain constraints.

1 Introduction and preliminaries

The aim of this note is to reveal deeper properties of maximal monotone operators that enjoy, under a locally convex space settings, the classical sum theorem which, in the literature, is sometimes called, when the context is provided by Banach spaces, the Rockafellar conjecture.

A breakthrough in the study of maximal monotone operators is represented by the introduction, in 2006 in [15, Theorem 2.3], of a new characterization of maximal monotonicity based on the notions of representability and “NI-type” operator (see [18, Remark 3.5] or Remark 11 below for more details). This characterization works in locally convex spaces and, is the main argument used after 2006 in the majority of the articles concerning the calculus rules for maximal monotone operators in general Banach spaces such as those in [13, 15, 16, 17, 18, 19, 20, 23].

The present paper enhances the aforementioned maximality characterization by presenting localized versions of it together with their direct consequences.

The plan of the paper is as follows. Section 2 presents the three main notions studied in this article together with their immediate properties and some variants. Section 3 is concerned with the interplay of these notions. Section 4 contains the representability of the sum of two representable operators. We conclude our article with some open problems in Section 5.

Throughout this paper, if not otherwise explicitly mentioned, (X, τ) is a non-trivial (that is, $X \neq \{0\}$) Hausdorff separated locally convex space (LCS for short), X^* is its topological dual endowed with the weak-star topology ω^* , the topological dual of (X^*, ω^*) is identified with X , and the weak topology on X is denoted by ω .

We denote by $\mathcal{V}_\tau(x)$ the family of τ -neighborhoods of $x \in X$ and the convergence of nets in (X, τ) by $x_i \xrightarrow{\tau} x$.

The *duality product* or *coupling* of $X \times X^*$ is denoted by $\langle x, x^* \rangle := x^*(x) =: c(x, x^*)$, for $x \in X$, $x^* \in X^*$. As usual, with respect to the dual system (X, X^*) , we denote the *orthogonal* of $S \subset X$ by $S^\perp := \{x^* \in X^* \mid \langle x, x^* \rangle = 0, \text{ for every } x \in S\}$ and the *support function* of S by $\sigma_S(x^*) := \sup_{x \in S} \langle x, x^* \rangle$, $x^* \in X^*$ while for $M \subset X^*$, the orthogonal of M is denoted by $M^\perp := \{x \in X \mid \langle x, x^* \rangle = 0, \text{ for every } x^* \in M\}$ and its support function is $\sigma_M(x) = \sup_{x^* \in M} \langle x, x^* \rangle$, $x \in X$.

To a multi-valued operator $T : X \rightrightarrows X^*$ we associate its

- *graph*: $\text{Graph } T = \{(x, x^*) \in X \times X^* \mid x^* \in Tx\}$,
- *inverse*: $T^{-1} : X^* \rightrightarrows X$, $\text{gph } T^{-1} = \{(x^*, x) \mid (x, x^*) \in \text{Graph } T\}$,
- *domain*: $D(T) := \{x \in X \mid Tx \neq \emptyset\} = \text{Pr}_X(\text{Graph } T)$, and
- *range*: $R(T) := \{x^* \in X^* \mid x^* \in Tx \text{ for some } x \in X\} = \text{Pr}_{X^*}(\text{Graph } T)$.

Here Pr_X and Pr_{X^*} are the projections of $X \times X^*$ onto X and X^* , respectively.

- *direct image*: $T(A) := \cup_{x \in A} Tx$, $A \subset X$.

When no confusion can occur, $T : X \rightrightarrows X^*$ will be identified with $\text{Graph } T \subset X \times X^*$.

In the sequel, given a locally convex space (E, τ) and $S \subset E$, the following notations are used: “ $\text{cl}_\tau S = \overline{S}^\tau$ ” for the τ -closure of S , “ $\text{int}_\tau S$ ” for the τ -topological interior of S , “ $\text{bd}_\tau S = \overline{S}^\tau \setminus \text{int}_\tau S$ ” for the boundary of S , “ $\text{conv } S$ ” for the *convex hull* of S , “ $\text{aff } S$ ” for the *affine hull* of S , and “ $S^i = \text{core } S$ ” for the *algebraic interior* of S , “ $^i S$ ” for the *relative algebraic interior* of S with respect to $\text{aff } S$. When the topology τ is implicitly understood the use of the τ -notation is avoided.

A set $S \subset E$ is called *algebraically open* if $S = \text{core } S$.

We denote by ι_S the *indicator function* of $S \subset E$ defined by $\iota_S(x) := 0$ for $x \in S$ and $\iota_S(x) := \infty$ for $x \in E \setminus S$.

The set $[x, y] := \{tx + (1-t)y \mid 0 \leq t \leq 1\} \subset E$ represents the closed segment with end-points $x, y \in E$.

For $f, g : E \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ we set $[f \leq g] := \{x \in E \mid f(x) \leq g(x)\}$, $[f = g]$, $[f < g]$ and $[f > g]$ are similarly defined, while e.g. $f \geq g$ means $[f \geq g] = E$ or, for every $e \in E$, $f(e) \geq g(e)$.

We consider the following classes of functions and operators on X :

$\Lambda(X)$ is the class of proper convex functions $f : X \rightarrow \overline{\mathbb{R}}$. Recall that f is *proper* if $\text{dom } f := \{x \in X \mid f(x) < \infty\}$ is nonempty and f does not take the value $-\infty$;

$\Gamma_\tau(X)$ is the class of functions $f \in \Lambda(X)$ that are τ -lower semicontinuous (τ -lsc for short); when the topology is implicitly understood the notation $\Gamma(X)$ is used instead;

$\mathcal{M}(X)$ is the class of non-void monotone operators $T : X \rightrightarrows X^*$ ($\text{Graph } T \neq \emptyset$). Recall that $T : X \rightrightarrows X^*$ is *monotone* if $\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0$ for all $x_1, x_2 \in D(T)$, $x_1^* \in Tx_1$, $x_2^* \in Tx_2$;

$\mathfrak{M}(X)$ is the class of *maximal monotone* operators $T : X \rightrightarrows X^*$. The maximality is understood in the sense of graph inclusion as subsets of $X \times X^*$.

Recall some notions associated to a proper function $f : X \rightarrow \overline{\mathbb{R}}$:

$\text{epi } f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is the *epigraph* of f ;

$\text{cl}_\tau f : X \rightarrow \overline{\mathbb{R}}$, the τ -lsc hull of f , is the greatest τ -lsc function majorized by f ; $\text{epi}(\text{cl}_\tau f) = \text{cl}_\tau(\text{epi } f)$;

$\text{conv } f : X \rightarrow \overline{\mathbb{R}}$, the *convex hull* of f , is the greatest convex function majorized by f ; $(\text{conv } f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \text{conv}(\text{epi } f)\}$ for $x \in X$;

$\overline{\text{conv}}^\tau f : X \rightarrow \overline{\mathbb{R}}$, the τ -lsc convex hull of f , is the greatest τ -lsc convex function majorized by f ; $\text{epi}(\overline{\text{conv}}^\tau f) := \overline{\text{conv}}^\tau(\text{epi } f)$;

$f^* : X^* \rightarrow \overline{\mathbb{R}}$ is the *convex conjugate* of $f : X \rightarrow \overline{\mathbb{R}}$ with respect to the dual system (X, X^*) , $f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) \mid x \in X\}$ for $x^* \in X^*$;

$\partial f(x)$ is the *subdifferential* of the proper function $f : X \rightarrow \overline{\mathbb{R}}$ at $x \in X$; $\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle + f(x) \leq f(x'), \forall x' \in X\}$ for $x \in X$ (it follows from its definition that $\partial f(x) := \emptyset$ for $x \notin \text{dom } f$). Recall that $N_C = \partial \iota_C$ is the *normal cone* to C , where $\iota_C(x) = 0$, if $x \in C$, $\iota_C(x) = +\infty$ otherwise; ι_C is the indicator function of $C \subset X$.

For (X, τ) a LCS, let $Z := X \times X^*$. It is known that $(Z, \tau \times \omega^*)^* = Z$ via the coupling

$$z \cdot z' := \langle x, x'^* \rangle + \langle x', x^* \rangle, \quad \text{for } z = (x, x^*), \quad z' = (x', x'^*) \in Z;$$

(Z, Z) is called the *natural dual system*.

For a proper function $f : Z \rightarrow \overline{\mathbb{R}}$ all the above notions are defined similarly. In addition, with respect to the natural dual system (Z, Z) , the conjugate of f is given by

$$f^\square : Z \rightarrow \overline{\mathbb{R}}, \quad f^\square(z) = \sup\{z \cdot z' - f(z') \mid z' \in Z\},$$

and by the biconjugate formula, $f^{\square\square} = \overline{\text{conv}}^{\tau \times \omega^*} f$ whenever f^\square (or $\overline{\text{conv}}^{\tau \times \omega^*} f$) is proper.

We introduce the following classes of functions:

$$\begin{aligned} \mathcal{C} &:= \mathcal{C}(Z) := \{f \in \Lambda(Z) \mid f \geq c\}, \\ \mathcal{R} &:= \mathcal{R}(Z) := \Gamma_{\tau \times \omega^*}(Z) \cap \mathcal{C}(Z), \\ \mathcal{D} &:= \mathcal{D}(Z) := \{f \in \mathcal{R}(Z) \mid f^\square \geq c\}. \end{aligned}$$

It is known that $[f = c] \in \mathcal{M}(X)$ for every $f \in \mathcal{C}(Z)$ (see e.g. [8, Proposition 4(h)], [18, Lemma 3.1]).

Lemma 1 *Let X be a LCS and let $h \in \mathcal{C}$. Then $[h = c] \subset [h^\square = c]$. If, in addition, $h \in \mathcal{D}$ then $[h = c] = [h^\square = c]$.*

Proof. Let $z \in [h = c]$. Then

$$h'(z; w) = \lim_{t \downarrow 0} \frac{h(z + tw) - h(z)}{t} \geq \lim_{t \downarrow 0} \frac{c(z + tw) - c(z)}{t} = z \cdot w, \quad \forall w \in Z,$$

which shows that $z \in \partial h(z)$. Therefore $h(z) + h^\square(z) = z \cdot z = 2c(z)$ and so $h^\square(z) = c(z)$.

If, in addition, $h \in \mathcal{D}$ the stated equality follows from $h = h^{\square\square}$ and the previously shown inclusion applied for h and h^\square . ■

To a multifunction $T : X \rightrightarrows X^*$ we associate the following functions: $c_T : Z \rightarrow \overline{\mathbb{R}}$, $c_T := c + \iota_{\text{Graph } T}$, $\psi_T : Z \rightarrow \overline{\mathbb{R}}$, $\psi_T := \text{cl}_{\tau \times \omega^*}(\text{conv } c_T)$, $\varphi_T : Z \rightarrow \overline{\mathbb{R}}$, $\varphi_T := c_T^\square = \psi_T^\square$ – the *Fitzpatrick function* of T . In expanded form

$$\varphi_T(z) := \varphi_T(x, x^*) := \sup\{z \cdot w - c(w) \mid w \in T\} = \sup\{\langle x, u^* \rangle + \langle u, x^* \rangle - \langle u, u^* \rangle \mid (u, u^*) \in T\},$$

for $z = (x, x^*) \in Z$. The function φ_T , ψ_T were introduced first in [4], [15].

Recall that whenever $T \in \mathcal{M}(X)$, $\varphi_T, \psi_T \in \Gamma_{\tau \times \omega^*}(Z)$.

The set $T^+ := [\varphi_T \leq c]$ describes all elements of Z that are *monotonically related* (m.r. for short) to T .

Let us recall several properties of these functions.

Theorem 2 *Let X be a LCS.*

- (i) *For every $T \subset X \times X^*$, $T \subset (D(T) \times X^*) \cup (X \times R(T)) \subset [\varphi_T \geq c]$ (see [15, Theorem 1.1], [14, 17, Proposition 3.2]),*
- (ii) *$T \in \mathcal{M}(X)$ iff $T \subset [\varphi_T = c]$ iff $\psi_T \geq c$, ([14, 17, Proposition 3.2], [23, (8)])*
- (iii) *$T \in \mathfrak{M}(X)$ iff $T \in \mathcal{M}(X)$, $T = [\psi_T = c]$, and $\varphi_T \geq c$ ([15, Theorems 2.2, 2.3], [23, Theorem 1]).*

For other properties of φ_T, ψ_T we refer to [15, 17, 18, 22, 23].

The following set properties are frequently used in the sequel; for $M, N \subset X \times X^*$, $V, W \subset X$

- $M \cap (V \times X^*) \subset N \Rightarrow (\text{Pr}_X M) \cap V \subset \text{Pr}_X N$;
- $M \cap (V \times X^*) \subset W \times X^* \Leftrightarrow (\text{Pr}_X M) \cap V \subset W$; and
- $\text{Pr}_X(M \cap (V \times X^*)) = (\text{Pr}_X M) \cap V$.

Throughout this article the conventions $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$ are observed.

2 Definitions and properties

Definition 3 Let X be a LCS. A subset $V \subset X$ *identifies* $T : X \rightrightarrows X^*$ or T is *identified by* V if $[\varphi_{T|_V} \leq c] \cap V \times X^* \subset \text{Graph } T$. Equivalently, V identifies T iff every $z = (x, x^*) \in V \times X^*$ that is m.r. to $T|_V$ belongs to T . Here $T|_V : X \rightrightarrows X^*$ is defined by $(x, x^*) \in T|_V$ if $x \in V$ and $(x, x^*) \in T$ or $\text{Graph } T|_V = \text{Graph } T \cap (V \times X^*)$.

Note that the empty set identifies any operator, but if a non-empty $V \subset X$ identifies T then $V \cap D(T) \neq \emptyset$. Indeed, if $V \neq \emptyset$ and $V \cap D(T) = \emptyset$ then $\varphi_{T|_V} = -\infty$, $[\varphi_{T|_V} \leq c] \cap V \times X^* = V \times X^* \not\subset \text{Graph } T$, that is, V does not identify T . Hence this notion is interesting only when $V \cap D(T) \neq \emptyset$.

When T is non-void monotone, V identifies T iff $T|_V$ is maximal monotone in $V \times X^*$, that is, $\text{Graph } T|_V$ has no proper monotone extension in $V \times X^*$ or $[\varphi_{T|_V} \leq c] \cap V \times X^* = \text{Graph}(T|_V)$.

In the context of a Banach space X , a monotone operator T is called of type (FPV) or maximal monotone locally (notion first introduced in [12] and further studied in [3]) if for every open convex $V \subset X$ either $V \cap D(T) = \emptyset$ or V identifies T . For the sake of language, notation simplicity, and notion uniformity we introduce the terminology *identifiable* as an extension of the type (FPV) notion to a general operator in the context of locally convex spaces.

Definition 4 Let (X, τ) be a LCS. An operator $T : X \rightrightarrows X^*$ is $(\tau-)$ *identifiable* if T is identified by every $(\tau-)$ open convex subset V of X such that $V \cap D(T) \neq \emptyset$.

Note that X identifies a monotone operator T iff T is maximal monotone. Therefore every identifiable monotone operator is maximal monotone.

The identifiability of a maximal monotone operator T is interesting only on sets V with $D(T) \not\subset V$ since $T \in \mathfrak{M}(X)$ is identified by every V that contains $D(T)$.

The τ -identifiability of an operator depends explicitly on the topology τ and not only on the duality (X, X^*) .

The identifiability notion unifies several other notions from the literature. For example for X a Banach space, $T : X \rightrightarrows X^*$ is *locally maximal monotone* (notion introduced in [2, p. 583]) iff T is monotone and $T^{-1} : X^* \rightrightarrows X$ is identified by every norm-open convex subset of X^* .

The identifiability of a monotone operator T is intrinsically related to the sum theorem. More precisely, if $T + N_C$ is maximal monotone, for every $C \subset X$ closed convex with $D(T) \cap \text{int } C \neq \emptyset$ then T is identifiable. Under a Banach space settings this implication is known for some time (see e.g. [3, Proposition 3.3]) but it also holds in a locally convex space context.

The sum conjecture [SC] is true in reflexive Banach spaces (see e.g. [9, Theorem 1(a), p. 76]). Therefore every maximal monotone operator in a reflexive Banach space is identifiable.

The class of open convex sets arises naturally in the identification of maximal monotone operators. That is not the case for the class of closed convex sets (even when they have non-empty interiors). Assume that X is a LCS and the closed convex $C \subset X$ identifies $T \in \mathcal{M}(X)$. Since $T|_C \subset T + N_C \in \mathcal{M}(X)$ and $D(T + N_C) \subset C$ one gets $T|_C = T + N_C \subset T$. The contrapositive form of this fact shows that a closed convex $C \subset X$ does not identify $T \in \mathcal{M}(X)$ if $T + N_C \in \mathfrak{M}(X)$ (which happens for example when X is a reflexive Banach space and $D(T) \cap \text{int } C \neq \emptyset$) and $D(T) \not\subset C$. However, in general, C can identify $T + N_C$ (see Theorem 27 below) and that leads to our next notion.

Definition 5 Let X be a LCS. A subset $V \subset X$ *locates* $T : X \rightrightarrows X^*$ in $S \subset X$ or T is *located by* V in S if $\text{Pr}_X[\varphi_{T|_V} \leq c] \cap V \subset S$ (or $[\varphi_{T|_V} \leq c] \cap V \times X^* \subset S \times X^*$). Equivalently, V locates T in S iff every $z = (x, x^*) \in V \times X^*$ that is m.r. to $T|_V$ has $x \in S$.

An operator $T : X \rightrightarrows X^*$ is called *locatable in* S iff every open convex subset V of X such that $V \cap D(T) \neq \emptyset$ locates T in S . When $S = D(T)$ we simply say that V *locates* T or T is *located by* V and that T is *locatable*.

As previously seen, his notion is interesting only when $V \cap D(T) \neq \emptyset$ because if $V \cap D(T) = \emptyset$ then $\Pr_X[\varphi_{T|_V} \leq c] \cap V \subset S$ reduces to $V \subset S$.

A monotone operator T is locatable if for every open convex set $V \subset X$ with $V \cap D(T) \neq \emptyset$, $T|_V$ cannot be extended outside $D(T) \cap V$, as a monotone operator in $V \times X^*$.

In the literature, for X a Banach space, an operator T is called of type weak-FPV (notion first introduced in [19]) if every open convex subset $V \subset X$ with $V \cap D(T) \neq \emptyset$ locates T . The terminology locatable is used as an extension and a simplified notation of the type weak-FPV notion to the general locally convex space settings.

Every locatable operator in S is locatable in S' , whenever $S \subset S'$. Also, every identifiable operator is locatable because V locates T whenever V identifies T . However, there exist monotone operators that are locatable but not identifiable. Take for example $T = \{0\} \times (X^* \setminus \{0\})$ where X is a LCS. Note that T is not identifiable since it is not maximal monotone. By a direct verification T is locatable. Indeed, if V is open convex with $0 \in V$ and $z = (x, x^*) \in V \times X^*$ is m.r. to $T|_V = T$ then $x = 0$ because $\{0\} \times X^*$ is the unique maximal monotone extension of T .

We will see later, in Theorem 23 below, that for a maximal monotone operator in the general context of a locally convex space the locatable and identifiable notions coincide.

An operator $T : X \rightrightarrows X^*$ is automatically located by every $V \subset D(T)$. Therefore the location of an operator T is interesting only on sets $V \not\subset D(T)$.

Lemma 6 *Let X be a LCS and let $T : X \rightrightarrows X^*$. Then X locates T iff $\varphi_T \geq c$ and $\Pr_X[\varphi_T = c] \subset D(T)$. If, in addition, $T \in \mathcal{M}(X)$ then X locates T iff $\varphi_T \geq c$ and $\Pr_X[\varphi_T = c] = D(T)$.*

Proof. Condition X locates T comes to $[\varphi_T \leq c] \subset D(T) \times X^*$. The conclusion follows after we take Theorem 2 (i), (ii) into account, that is, for every $T : X \rightrightarrows X^*$, $D(T) \times X^* \subset [\varphi_T \geq c]$ and that $T \subset [\varphi_T = c]$ whenever $T \in \mathcal{M}(X)$. ■

Therefore the localization of an operator T by X depends on the condition $\varphi_T \geq c$ also known as T is of type negative-infimum (NI for short); notion that was first used in [15] and introduced in [14, 18, Remark. 3.5]. Therefore, according to Lemma 6, every maximal monotone operator is NI because $T \in \mathfrak{M}(X) \Leftrightarrow X$ identifies $T \Rightarrow X$ locates T .

It must be said that our NI notion differs fundamentally from the NI notion introduced, for X a Banach space, in $X^* \times X^{**}$ by Simons (see e.g. [11, Definition 25.5, p. 99]). The NI-operators in the sense of Simons coincide with those of dense-type in the sense of Gossez introduced in [5] (see [7]). Since the dense-type property is stronger and has been introduced prior to the NI class in the sense of Simons it is our opinion that the use of NI notion in the sense of Simons is obsolete. Another essential difference between these two notions, besides the underlying space context, is that every maximal monotone operator is NI in the current sense while not every maximal monotone operator is of dense-type (see e.g. [6, p. 89]). For more explanations on comparing these notions see [22, p. 33] and [19, p. 662].

Since every X -locatable operator is NI, we expect in general that the localization property depend on a localized NI type condition.

Definition 7 Let X be a LCS. An operator $T : X \rightrightarrows X^*$ is of *negative-infimum type* on $V \subset X$ or simply V -NI if $V \times X^* \subset [\varphi_{T|_V} \geq c]$. Equivalently, T is V -NI iff $[\varphi_{T|_V} < c] \cap V \times X^* = \emptyset$ iff $\Pr_X[\varphi_{T|_V} < c] \cap V = \emptyset$.

The operator $T : X \rightrightarrows X^*$ is called *locally-NI* if, for every open convex $V \subset X$ such that $V \cap D(T) \neq \emptyset$, T is V -NI.

Note that every operator is \emptyset -NI and if, for a certain $V \neq \emptyset$, T is V -NI then $V \cap D(T) \neq \emptyset$, because $V \cap D(T) = \emptyset$ implies $\varphi_{T|_V} = -\infty$. Notice also that if T is V -NI then $V \subset \Pr_X[\varphi_{T|_V} \geq c]$ while the converse is not true. The X -NI type coincides with the NI type discussed above.

Theorem 8 Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be such that $V \cap D(T) \neq \emptyset$. The following are equivalent

- (i) T is V -NI,
- (ii) $\varphi_{T|_V} \geq c$ in $V \times X^*$,
- (iii) $\varphi_{T|_V} + \iota_{V \times X^*} \geq c$,
- (iv) $[\varphi_{T|_V} \leq c] \cap V \times X^* \subset [\varphi_{T|_V} \geq c]$,
- (v) $[\varphi_{T|_V} < c] \cap V \times X^* \subset [\varphi_{T|_V} \geq c]$.

Proof. Left to the reader. ■

Theorem 9 Let X be a LCS and let $T \in \mathcal{M}(X)$. Assume that

$$V \text{ open convex, } V \cap D(T) \neq \emptyset \implies \text{Pr}_X[\varphi_{T|_V} < c] \cap V \subset \overline{D(T)}. \quad (1)$$

Then $\overline{D(T)}$ is convex.

In particular, if $T \in \mathcal{M}(X)$ is locatable (in $\overline{D(T)}$) or $T \in \mathcal{M}(X)$ is locally-NI then $\overline{D(T)}$ is convex.

Proof. Assume that $\overline{D(T)}$ is not convex. There exist $x_0, x_1 \in D(T)$, $0 < \rho < 1$, and $U \in \mathcal{V}(0)$ such that $(x_\rho + U) \cap D(T) = \emptyset$, where $x_t := tx_1 + (1-t)x_0$, $0 \leq t \leq 1$; in particular $x_0 \neq x_1$. Let $x_0^* \in Tx_0$, $x_1^* \in Tx_1$, and denote by $z_0 = (x_0, x_0^*)$, $z_1 = (x_1, x_1^*)$, $z_\rho := \rho z_1 + (1-\rho)z_0$.

Let $V \in \mathcal{V}(0)$ be balanced and $\gamma > 0$ be such that $V + V \subset U$ and $x_1 - x_0 \in \gamma V$. Take $y^* \in X^*$ with $\langle x_0 - x_1, y^* \rangle \geq \gamma(\rho(1-\rho)c(z_1 - z_0) + 1) > 0$ and $W \in \mathcal{V}(0)$ be open convex such that $W \subset V$ and $\alpha := \sup\{|\langle u, y^* \rangle| \mid u \in W\}$ is finite and positive (see e.g. [10, Theorem 1.18, p. 15]). Let $0 < \epsilon < \min\{1, 1/\alpha\}$. Let $\bar{z}_\rho := z_\rho + (0, y^*)$.

For every $z = (x, x^*) \in T$ with $x \in D(T) \cap ([x_1, x_\rho] + \epsilon W)$ we have $x - x_\rho \notin U$ and $x - x_\lambda \in \epsilon W$ for some $1 \geq \lambda > \rho$. This yields that $x_\lambda - x_\rho = (\lambda - \rho)(x_1 - x_0) \notin V$. Hence $(\lambda - \rho) \geq 1/\gamma$. Since $x_\rho - x = (\lambda - \rho)(x_0 - x_1) + x_\lambda - x$ and $T \in \mathcal{M}(X)$ we have

$$\begin{aligned} c(\bar{z}_\rho - z) &= c(z_\rho - z) + \langle x_\rho - x, y^* \rangle = \rho c(z_1 - z) + (1-\rho)c(z_0 - z) - \rho(1-\rho)c(z_1 - z_0) + \langle x_\rho - x, y^* \rangle \\ &\geq (\lambda - \rho)\langle x_0 - x_1, y^* \rangle + \langle x_\lambda - x, y^* \rangle - \rho(1-\rho)c(z_1 - z_0) \geq \frac{1}{\gamma}\langle x_0 - x_1, y^* \rangle - \alpha\epsilon - \rho(1-\rho)c(z_1 - z_0) > 0. \end{aligned}$$

This yields $\bar{z}_\rho \in [\varphi_{T|_{[x_1, x_\rho] + \epsilon W}} < c] \cap ([x_1, x_\rho] + \epsilon W) \subset \overline{D(T)} \times X^*$ and the contradiction $x_\rho \in \overline{D(T)}$. ■

Remark 10 (T is V -NI versus $T|_V$ is NI) First note that T is V -NI whenever $V \subset D(T)$ since in this case $V \times X^* = D(T|_V) \times X^* \subset [\varphi_{T|_V} \geq c]$ (see Theorem 2 (i)) or because, in this case, V locates T (see Theorem 13 below). Also, it is straightforward that T is V -NI whenever $T|_V$ is $(X-)$ NI. The converse of this fact, namely, whether $T|_V$ is NI whenever T is V -NI fails to be true in any LCS X even when T is maximal monotone, V is convex, and V is open or closed with empty or non-empty interior.

We base our following examples on the fact that every monotone NI operator admits a unique maximal monotone extension (see Theorem 22 below or [22, Proposition 4 (iii)]).

For example, for every $T \in \mathcal{M}(X)$ with a non-singleton domain and for every $x \in D(T)$, T is $\{x\}$ -NI while $T|_{\{x\}} = \{x\} \times Tx$ is not NI because $\{x\} \times X$ and any maximal monotone extension of T are different maximal monotone extensions of $T|_{\{x\}}$.

Similar considerations can be made for a non-NI operator T which is $D(T)$ -NI but $T|_{D(T)} = T$ is not NI.

Let $C \subsetneq X$ be closed convex with $\text{int } C \neq \emptyset$. Then N_C is $\text{int } C$ -NI (this fact can also be checked directly from $N_C|_{\text{int } C} = \text{int } C \times \{0\}$ and $\varphi_{\text{int } C \times \{0\}}(x, x^*) = \sigma_C(x^*)$, $(x, x^*) \in X \times X^*$). But $N_C|_{\text{int } C}$ is not NI since it admits two distinct (maximal) monotone extensions: N_C and $X \times \{0\}$. Similarly, for every closed convex set $D \subset \text{int } C$ (with possible empty interior) we have that N_C is D -NI and $N_C|_D$ is not NI because $N_C|_{\text{int } C}$ is not NI.

Remark 11 (The NI method) In general it is hard to verify the NI condition directly, even when T is monotone, since the closed forms of φ_T , ψ_T are known only for few types of operators (see e.g. [1, 22]) and, when X is a non-reflexive Banach space, the coupling c is not continuous with respect to any topology on $X \times X^*$ compatible with the natural duality $(X \times X^*, X^* \times X)$ (see [23, Appendix]).

Given a LCS X , the first direct method to prove that an operator $T : X \rightrightarrows X^*$ is of NI type has been developed in [15, Theorem. 1.1] and is summarized as follows

$$(z = (x, x^*) \text{ is m.r. to } T \Rightarrow x \in D(T)) \implies T \text{ is NI}, \quad (2)$$

or, equivalently, X locates $T \implies T$ is NI. The reader recognizes that this NI method is contained in Lemma 6 and that its converse holds under the additional condition $\Pr_X[\varphi_T = c] \subset D(T)$.

The following is a slightly improved version of (2), namely

$$(\Pr_X[\varphi_T < c] \subset D(T)) \implies T \text{ is NI}. \quad (3)$$

Indeed, if $z = (x, x^*) \in [\varphi_T < c]$ then $x \in \Pr_X[\varphi_T < c] \subset D(T)$; whence, according to Theorem 2 (i), $z \in [\varphi_T \geq c]$ a contradiction. Therefore $[\varphi_T < c]$ is empty, that is, T is NI.

Similar considerations for a V -NI method are contained in the following result.

Theorem 12 (The V -NI method) *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$. Then T is V -NI iff $\Pr_X[\varphi_{T|_V} < c] \cap V \subset D(T)$.*

Proof. It suffices to note that, in general, $\Pr_X[\varphi_{T|_V} < c] \cap V \cap D(T) = \emptyset$ due to $D(T|_V) \times X^* = (D(T) \cap V) \times X^* \subset [\varphi_{T|_V} \geq c]$. ■

Theorem 13 *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be such that $V \cap D(T) \neq \emptyset$. The following are equivalent*

- (i) V locates T ,
- (ii) T is V -NI and $\Pr_X[\varphi_{T|_V} = c] \cap V \subset D(T)$,
- (iii) $\Pr_X \text{ dom } \varphi_{T|_V} \cap V \subset \Pr_X([\varphi_{T|_V} \geq c] \cap \text{dom } \varphi_{T|_V})$ and $\Pr_X[\varphi_{T|_V} = c] \cap V \subset D(T)$.

If, in addition, $T|_V \in \mathcal{M}(X)$ then V locates T iff T is V -NI and $\Pr_X[\varphi_{T|_V} = c] \cap V = D(T) \cap V$.

Proof. (i) \Rightarrow (ii) Recall that V locates T means $[\varphi_{T|_V} \leq c] \cap V \times X^* \subset D(T) \times X^*$ from which $[\varphi_{T|_V} \leq c] \cap V \times X^* \subset D(T|_V) \times X^* \subset [\varphi_{T|_V} \geq c]$; whence T is V -NI and $\Pr_X[\varphi_{T|_V} \leq c] \cap V = \Pr_X[\varphi_{T|_V} = c] \cap V \subset D(T)$.

(ii) \Rightarrow (iii) Because T is V -NI we have $V \times X^* \subset [\varphi_{T|_V} \geq c]$ followed by $\text{dom } \varphi_{T|_V} \cap V \times X^* \subset \text{dom } \varphi_{T|_V} \cap [\varphi_{T|_V} \geq c]$.

(iii) \Rightarrow (i) Let $x \in \Pr_X[\varphi_{T|_V} \leq c] \cap V$. Take $x_1^* \in X^*$ such that $(x, x_1^*) \in [\varphi_{T|_V} \leq c] \subset \text{dom } \varphi_{T|_V}$. Then $x \in \Pr_X(\text{dom } \varphi_{T|_V}) \cap V \subset \Pr_X([\varphi_{T|_V} \geq c] \cap \text{dom } \varphi_{T|_V})$, so $(x, x_2^*) \in [\varphi_{T|_V} \geq c] \cap \text{dom } \varphi_{T|_V}$ for some $x_2^* \in X^*$. The function $f : [0, 1] \rightarrow \mathbb{R}$, $f(t) = (\varphi_{T|_V} - c)(x, tx_1^* + (1-t)x_2^*)$ is continuous and $f(0) \geq 0$, $f(1) \leq 0$. Therefore there is $s \in [0, 1]$ such that $f(s) = 0$, that is, $(x, sx_1^* + (1-s)x_2^*) \in [\varphi_{T|_V} = c]$. Therefore $x \in \Pr_X[\varphi_{T|_V} = c] \cap V \subset D(T)$ and so V locates T .

If in addition $T|_V \in \mathcal{M}(X)$ then $T|_V \subset [\varphi_{T|_V} = c]$, $D(T) \cap V = D(T|_V) \subset \Pr_X[\varphi_{T|_V} = c]$, and the last part of the conclusion follows from (i) \Leftrightarrow (ii). ■

In Theorem 13 we saw that T being of V -NI type is an important part of V locating T and as a consequence every locatable operator is locally-NI. As previously stated, the other condition in Theorem 13, namely, $\Pr_X[\varphi_{T|_V} = c] \cap V \subset D(T)$ is hard to verify directly due to the unwieldy nature of φ_T . Fortunately, this latter condition can be replaced by representability.

Definition 14 Let (X, τ) be a LCS. An operator $T : X \rightrightarrows X^*$ is *representable in* $V \subset X$ or *V-representable* if $V \cap D(T) \neq \emptyset$ and there is $h \in \mathcal{R}$ (that is, $h \geq c$ and $h \in \Gamma_{\tau \times w^*}(X \times X^*)$) such that $[h = c] \cap V \times X^* = \text{Graph}(T|_V)$. The function h is called a *V-representative of T*. The class of V-representatives of T is denoted by \mathcal{R}_T^V .

As previously seen, the condition $V \cap D(T) \neq \emptyset$ can be avoided but its presence makes the previous definition meaningful.

An X -representable operator $T : X \rightrightarrows X^*$ is simply called *representable* and the class of its representatives is denoted by \mathcal{R}_T , notion that was first considered in this form in [15]. For properties of representable operators see [17, 18, 21, 22, 23].

In other words, T is V -representable if $T|_V$ is the trace of the representable operator $[h = c]$ on $V \times X^*$, where $h \in \mathcal{R}$.

Remark 15 Note that

- $T|_V$ is monotone whenever T is V -representable since $[h = c] \in \mathcal{M}(X)$ for every $h \in \mathcal{C}$ (see e.g. [8, Proposition 4] or [18, Lemma 3.1]);
- T is W -representable whenever T is V -representable and $V \supset W$; in this case every V -representative is a W -representative of T , that is, $\mathcal{R}_T^V \subset \mathcal{R}_T^W$. In particular, if T is representable then, for every $V \subset X$, T is V -representable. Conversely, if T is V -representable then T need not be representable because we can modify T outside $V \times X^*$. For example, for every $V \subsetneq X$, $x \notin V$, $T = (X \setminus \{x\}) \times \{0\}$ is V -representable since $\varphi_T = \psi_T = \iota_{X \times \{0\}} \in \mathcal{R}_T^V$ and T is not representable because T is not closed or because $T \subsetneq [\psi_T = c]$ (see [15, Theorem 2.2] or Theorem 16 below);
- If $T|_V$ is representable then T is V -representable; in this case every representative of $T|_V$ is a V -representative of T , i.e., $\mathcal{R}_{T|_V} \subset \mathcal{R}_T^V$. Indeed, if $h \in \mathcal{R}$ has $T|_V = [h = c]$ then $V \times X^* \cap [h = c] = T|_V$. Conversely, if T is V -representable with h a V -representative of T such that $\text{Pr}_X[h = c] \subset V$ then $T|_V$ is representable with representative h . In general, without the additional condition, the converse is not true, in any LCS X , even if we work with $T \in \mathfrak{M}(X)$ and V open convex. Indeed, take $C \subsetneq X$ closed convex with $\text{int } C \neq \emptyset$, $T = N_C$, $V = \text{int } C$. Then T is representable, since it is maximal monotone (see e.g. [15, Theorem 2.3] or Theorem 20 below), while $T|_V = \text{int } C \times \{0\}$ is not, for example because $\psi_{T|_V} = \iota_{C \times \{0\}}$ and $T|_V = \text{int } C \times \{0\} \subsetneq C \times \{0\} = [\psi_{T|_V} = c]$ (see [15, Theorem 2.2] or Theorem 16 below);
- However, when C is closed convex, T is C -representable iff $T|_C$ is representable. Indeed, if h is a C -representative of T then $h + \iota_{C \times X^*}$ is a representative of $T|_C$.
- An operator $T \in \mathcal{M}(X)$ is $D(T)$ -representable whenever $D(T)$ identifies T . Indeed, let \bar{T} be a representable extension of T (such as $[\psi_T = c]$), and let $h \in \mathcal{R}_{\bar{T}}$, in particular, $\bar{T} = [h = c]$. Hence $[h = c] \cap D(T) \times X^* = \bar{T}|_{D(T)} = T$ because T is maximal monotone in $D(T) \times X^*$.

The following result is a generalization of [15, Theorem 2.2], [23, Theorem 1(ii)].

Theorem 16 Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be such that $V \cap D(T) \neq \emptyset$. The following are equivalent

- (i) T is V -representable,
- (ii) $T|_V \in \mathcal{M}(X)$ and $[\psi_{T|_V} = c] \cap V \times X^* \subset T|_V$,
- (iii) $T|_V \in \mathcal{M}(X)$ and $[\psi_{T|_V} = c] \cap V \times X^* = T|_V$,
- (iv) $\psi_{T|_V}$ is a V -representative of $T|_V$, i.e., $\psi_{T|_V} \in \mathcal{R}_{T|_V}^V$.

Proof. (i) \Rightarrow (ii) Let $h \in \mathcal{R}$ be such that $[h = c] \cap V \times X^* = T|_V$. Then $h \leq c_{T|_V}$ followed by $c \leq h \leq \psi_{T|_V}$ since $h \in \mathcal{R}$. Therefore $[\psi_{T|_V} = c] \subset [h = c]$ and so $[\psi_{T|_V} = c] \cap V \times X^* \subset T|_V$.

(ii) \Rightarrow (iii) From $T|_V \in \mathcal{M}(X)$ we know that $T|_V \subset [\psi_{T|_V} = c]$ (see [14, 17, Proposition 3.2 (viii)] or [23, (9)]).

For (iii) \Rightarrow (iv) it suffices to notice that $\psi_{T|_V} \in \mathcal{R}$, since $T|_V \in \mathcal{M}(X)$ (see Theorem 2(ii)).

The implication (iv) \Rightarrow (i) is trivial. ■

Remark 17 In case V is closed convex we have $\text{dom } \psi_{T|_V} \subset V \times X^*$ so $[\psi_{T|_V} = c] \subset V \times X^*$, $[\psi_{T|_V} = c] \cap V \times X^* = [\psi_{T|_V} = c]$, and Theorem 16 says again that T is V -representable iff $T|_V$ is representable.

Remark 18 Given X a LCS, $T : X \rightrightarrows X^*$, and $V \subset X$ such that $T|_V \in \mathcal{M}(X)$, the operator $R := [\psi_{T|_V} = c] \cap V \times X^*$ is the smallest V -representable extension of $T|_V$ in $V \times X^*$. Indeed, for every $h \in \mathcal{R}$ such that $[h = c] \cap V \times X^* \supset T|_V$ we have $h \leq c_{T|_V}$, $c \leq h \leq \psi_{T|_V}$ and so $[\psi_{T|_V} = c] \cap V \times X^* \subset [h = c] \cap V \times X^*$.

Also, $\varphi_R = \varphi_{T|_V}$, $\psi_R = \psi_{T|_V}$ because $T|_V \subset R \subset [\psi_{T|_V} = c]$ and $\varphi_{T|_V} = \varphi_{[\psi_{T|_V} = c]}$ (see [22, Proposition 4, p. 35]).

3 Characterizations

Theorem 19 Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be such that $V \cap D(T) \neq \emptyset$. Consider the conditions

- (i) $T|_V \in \mathcal{M}(X)$ and V identifies T ,
- (ii) T is V -representable and V locates T ,
- (iii) T is V -representable and V -NI.

Then (i) \Rightarrow (ii) \Rightarrow (iii). If, in addition, V is algebraically open then (i) \Leftrightarrow (ii) \Leftrightarrow (iii).

Proof. We adapt the proof in [15, Theorem 2.3] and we refer to [17, 18, 15] for other different arguments. The implication (ii) \Rightarrow (iii) is contained in Theorem 13.

(i) \Rightarrow (ii) Since V identifies it also locates T . Hence, according to Theorem 13, T is V -NI. We know that $\psi_{T|_V} \geq \max\{\varphi_{T|_V}, c\}$ since $T|_V \in \mathcal{M}(X)$ (see [14, 17, Proposition 3.2 (vii)]). This yields that $[\psi_{T|_V} = c] \cap V \times X^* \subset [\varphi_{T|_V} = c] \cap V \times X^* \subset T$ from which, according to Theorem 16, it follows that T is V -representable.

Assume that V is algebraically open, i.e., $V = \text{core } V$.

(iii) \Rightarrow (i) Since T is V -NI we have $[\varphi_{T|_V} \leq c] \cap V \times X^* = [\varphi_{T|_V} = c] \cap V \times X^*$ and from T being V -representable we know that $T|_V \in \mathcal{M}(X)$ so, according to Theorem 2 (ii), $\psi_{T|_V} \geq c$; whence $[\psi_{T|_V} = c] \subset [\varphi_{T|_V} = c]$ (see Lemma 1). This yields $T|_V = [\psi_{T|_V} = c] \cap V \times X^* \subset [\varphi_{T|_V} = c] \cap V \times X^*$ since T is V -representable. To conclude it suffices to show that

$$[\varphi_{T|_V} = c] \cap V \times X^* \subset [\psi_{T|_V} = c] \cap V \times X^*.$$

Let $z \in [\varphi_{T|_V} = c] \cap V \times X^*$. Because V is algebraically open, for every $v \in X \times X^*$ there is $t_v > 0$ such that $z + tv \in V \times X^*$, for every $0 < t < t_v$. Hence, since T is V -NI, $\varphi_{T|_V}(z + tv) \geq c(z + tv)$, for every $0 < t < t_v$. The directional derivative of $\varphi_{T|_V}$ at z in the direction of v satisfies

$$\forall v \in Z, \varphi'_{T|_V}(z; v) := \lim_{t \downarrow 0} \frac{\varphi_{T|_V}(z + tv) - \varphi_{T|_V}(z)}{t} \geq \lim_{t \downarrow 0} \frac{c(z + tv) - c(z)}{t} = z \cdot v.$$

This shows that $z \in \partial \varphi_{T|_V}(z)$, where “ ∂ ” is considered under the natural duality (Z, Z) . Therefore $\psi_{T|_V}(z) + \varphi_{T|_V}(z) = z \cdot z = 2c(z)$ which, together with $\varphi_{T|_V}(z) = c(z)$, implies that $\psi_{T|_V}(z) = c(z)$. ■

In particular, for $V = X$ we recover the following maximal monotonicity characterization.

Theorem 20 ([15, Theorem 2.3], [23, Theorem 1 (ii)]) *Let X be a LCS. Then $T \in \mathfrak{M}(X)$ iff T is representable and NI.*

Theorem 21 *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let \mathcal{V} be a class of algebraically open subsets of X such that $X \in \mathcal{V}$. Then $T \in \mathcal{M}(X)$ and T is identified by every $V \in \mathcal{V}$ iff T is representable and, for every $V \in \mathcal{V}$, T is V -NI.*

Proof. Since every representable operator is V -representable (for every $V \subset X$) and monotone the converse implication is straightforward from Theorem 19. For the direct implication one gets that T is maximal monotone (and implicitly representable) because it is identified by $X \in \mathcal{V}$. Again Theorem 19 completes the argument. ■

Theorem 22 *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be non-empty algebraically open and convex such that $T|_V \in \mathcal{M}(X)$ and T is V -NI. Then*

$$[\psi_{T|_V} = c] \cap V \times X^* = [\varphi_{T|_V} = c] \cap V \times X^* = [\varphi_{T|_V} \leq c] \cap V \times X^* \quad (4)$$

is the unique V -representable extension and the unique maximal monotone extension in $V \times X^$ of $T|_V$.*

If, in addition, $T \in \mathcal{M}(X)$ then the string of equalities in (4) can be completed to

$$[\psi_{T|_V} = c] \cap V \times X^* = [\varphi_T = c] \cap V \times X^* = [\psi_T = c] \cap V \times X^*. \quad (5)$$

If, in addition, $T \in \mathcal{M}(X)$ and T is V -representable then V identifies T and

$$\begin{aligned} [\psi_{T|_V} = c] \cap V \times X^* &= [\varphi_{T|_V} = c] \cap V \times X^* = [\varphi_{T|_V} \leq c] \cap V \times X^* \\ &= [\varphi_T = c] \cap V \times X^* = [\psi_T = c] \cap V \times X^* = \text{Graph}(T|_V). \end{aligned} \quad (6)$$

Proof. Let $R := [\psi_{T|_V} = c] \cap V \times X^*$. Then R is V -representable and V -NI since $T|_V \in \mathcal{M}(X)$, T is V -NI, and $\varphi_R = \varphi_{T|_V}$. According to Theorem 19, V identifies R , i.e., R is maximal monotone in $V \times X^*$. From $\psi_{T|_V} \geq \varphi_{T|_V}$ and $\varphi_{T|_V} \geq c$ in $V \times X^*$ we know that $R \subset [\varphi_{T|_V} = c] \cap V \times X^* = [\varphi_{T|_V} \leq c] \cap V \times X^*$ so $R = [\varphi_{T|_V} = c] \cap V \times X^*$ since $[\varphi_{T|_V} = c] \cap V \times X^* \in \mathcal{M}(X)$. Taking into consideration that R is the smallest V -representable extension of $T|_V$ the conclusion follows.

If, in addition, $T \in \mathcal{M}(X)$ then, due to the facts that T is V -NI and $T|_V \subset T$, we have that for every $z \in V \times X^*$

$$c(z) \leq \varphi_{T|_V}(z) \leq \varphi_T(z) \leq \psi_T(z) \leq \psi_{T|_V}(z),$$

whence $[\psi_{T|_V} = c] \cap V \times X^* \subset [\psi_T = c] \cap V \times X^* \subset [\varphi_T = c] \cap V \times X^* \subset [\varphi_{T|_V} = c] \cap V \times X^*$. Relation (4) completes the proof of (5).

Relation (6) follows from Theorems 16, 19 and relations (4), (5). ■

We are ready to prove that for a representable (and implicitly for a maximal monotone) operator the locatable and identifiable notions coincide.

Theorem 23 *Let X be a LCS and let $T : X \rightrightarrows X^*$. The following are equivalent*

- (i) $T \in \mathcal{M}(X)$ and T is identifiable,
- (ii) T is representable and locatable,
- (iii) T is representable and locally-NI.

In particular, every representable and locatable operator is maximal monotone.

Proof. (i) \Leftrightarrow (iii) is a particular case of Theorem 21 for $\mathcal{V} = \{V \subset X \mid V \text{ is open and convex, } V \cap D(T) \neq \emptyset\}$.

- (i) \Rightarrow (ii) is true since every identifiable monotone operator is locatable and maximal monotone.
- (ii) \Rightarrow (iii) is straightforward since every locatable operator is locally-NI. ■

The global representability condition in the previous theorem can be replaced by a weaker local form of it.

Definition 24 Let X be a LCS. An operator $T : X \rightrightarrows X^*$ is *low-representable* if, for every $z = (x, x^*) \in [\psi_T = c]$, there is $V \in \mathcal{V}(x)$ such that T is V -representable.

Every representable operator is low-representable (just take $V = X$).

Theorem 25 Let X be a LCS and let $T : X \rightrightarrows X^*$. Consider the conditions

- (iv) T is monotone, low-representable, and locatable;
- (v) T is monotone, low-representable, and locally-NI.

Conditions (i) – (iii) being those from Theorem 23, we have (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v).

Proof. The implications (ii) \Rightarrow (iv), (iii) \Rightarrow (v), (iv) \Rightarrow (v) are plain.

For (v) \Rightarrow (iii) we prove that T is representable, i.e., $[\psi_T = c] \subset T$. For every $z = (x, x^*) \in [\psi_T = c]$ let $V \in \mathcal{V}(x)$ be open convex and such that T is V -representable. Then, according to Theorem 22, $z \in [\psi_T = c] \cap V \times X^* \subset T$. ■

Theorem 26 Let (X, τ) be a LCS and let $T \in \mathcal{M}(X)$ be locally-NI. Then $[\varphi_T = c] = [\varphi_T \leq c] = [\psi_T = c]$ is the unique identifiable extension of T .

Proof. Because T is NI, from (4), $S := [\varphi_T = c] = [\varphi_T \leq c] = [\psi_T = c]$ is the unique maximal monotone extension of T . Since every identifiable operator is maximal monotone, it suffices to prove that S is locally-NI to get that S is the unique identifiable extension of T .

But, if an open convex $V \subset X$ has $V \cap D(S) \neq \emptyset$ then $V \cap D(T) \neq \emptyset$.

Indeed, if $x \in V \cap D(S)$ pick any $x^* \in S(x)$ and set $z = (x, x^*) \in S$. Then, since T is V -NI, $c(z) \leq \varphi_{T|_V}(z) \leq \varphi_T(z) = c(z)$ so, according to Theorem 22, $z \in [\varphi_{T|_V} = c] \cap V \times X^* = [\psi_{T|_V} = c] \cap V \times X^* \subset \text{dom } \psi_{T|_V} \subset \text{cl}_{\tau \times w^*}(\text{Graph}(T|_V))$ followed by $x \in \text{Pr}_X(\text{dom } \psi_{T|_V}) \subset \overline{\text{conv}}(D(T) \cap V) \subset \overline{D(T)}$ due to the convexity of $\overline{D(T)}$ (see Theorem 9). Hence $x \in V \cap \overline{D(T)} \neq \emptyset$ from which $V \cap D(T) \neq \emptyset$ since V is open.

Hence, for every open convex $V \subset X$ such that $V \cap D(S) \neq \emptyset$, $\varphi_{S|_V} \geq \varphi_{T|_V} \geq c$ in $V \times X^*$, because T is V -NI, i.e., S is V -NI. Therefore S is locally-NI. ■

The next result is a version of Theorem 19 for closed convex sets with non-empty interior. First note that for every $T : X \rightrightarrows X^*$ and $C \subset X$ we have

$$[\varphi_{T+N_C} \leq c] \cap C \times X^* = [\varphi_{T|_C} \leq c] \cap C \times X^*. \quad (7)$$

Indeed, the direct inclusion follows from $T|_C \subset T + N_C$. Conversely, if $z = (x, x^*) \in C \times X^*$ is m.r. to $T|_C$ and $(a, a^*) \in T|_C$, $n^* \in N_C(a)$ then $\langle x - a, n^* \rangle \leq 0$, $\langle x - a, x^* - a^* \rangle \geq 0$, and $\langle x - a, x^* - a^* - n^* \rangle \geq 0$, that is, z is m.r. to $T + N_C$.

Theorem 27 Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $C \subset X$ be closed convex such that $D(T) \cap \text{int } C \neq \emptyset$. If T is C -representable then the following are equivalent

- (i) C locates T ,
- (ii) T is C -NI,
- (iii) $[\varphi_{T|_C} \leq c] \cap C \times X^* \subset T + N_C$,
- (iv) C identifies $T + N_C$.

Proof. The implication (i) \Rightarrow (ii) is part of Theorem 13 while (iii) \Rightarrow (i) is plain.

(ii) \Rightarrow (iii) Since T is C -NI, we have $h := \varphi_{T|_C} + \iota_{C \times X^*} \in \mathcal{R}$ and $[h = c] = [\varphi_{T|_C} = c] \cap C \times X^* = [\varphi_{T|_C} \leq c] \cap C \times X^*$. According to [24, Theorem 2.8.7 (iii), p. 127]

$$h^\square(x, x^*) = \min\{\psi_{T|_C}(x, u^*) + \sigma_C(x^* - u^*) \mid u^* \in X^*\}, \quad (x, x^*) \in X \times X^*. \quad (8)$$

Here “min” stands for an infimum that is attained when finite.

For every $z = (x, x^*) \in [h^\square = c] \cap C \times X^*$ there is $v^* \in X^*$ such that $\psi_{T|_C}(x, v^*) + \sigma_C(x^* - v^*) = \langle x, x^* \rangle$. Since $\psi_{T|_C} \geq c$, $\sigma_C(x^* - v^*) \geq \langle x, x^* - v^* \rangle$, this implies that $(x, v^*) \in [\psi_{T|_C} = c] \cap C \times X^* = T|_C$, $x^* - v^* \in N_C(x)$, and $z \in T + N_C$. Therefore $[h^\square = c] \cap C \times X^* \subset T + N_C$. The inclusion $[h = c] \subset [h^\square = c]$ completes the proof of this implication (see Lemma 1).

(iii) \Leftrightarrow (iv) Since $(T + N_C)|_C = T + N_C$, this equivalence follows from (7). ■

In the absence of the C -representability of T the previous result still holds with T replaced by $R = [\psi_{T|_C} = c] \cap C \times X^*$ which is the smallest C -representable extension of T in the following string of implications: C locates $T \Rightarrow T$ is C -NI $\Leftrightarrow R$ is C -NI $\Leftrightarrow C$ locates $R \Leftrightarrow C$ identifies $R + N_C$. The converse of the first implication is false as seen from Remark 31 below for $C = \overline{V}$.

Corollary 28 *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $C \subset X$ be closed convex such that $D(T) \cap \text{int } C \neq \emptyset$ and T is C -representable. Then $T + N_C \in \mathfrak{M}(X)$ iff C locates T and $[\varphi_{T+N_C} \leq c] \subset C \times X^*$ iff T is C -NI and $[\varphi_{T+N_C} \leq c] \subset C \times X^*$.*

Proposition 29 *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be open convex such that $D(T) \cap V \neq \emptyset$ and $T|_V \in \mathcal{M}(X)$. If T is V -NI then*

$$\varphi_{T|_V} \geq c, \text{ on } \overline{V} \times X^*.$$

In particular, for every $V \subset S \subset \overline{V}$, T is S -NI.

If, in addition, T is \overline{V} -representable, then

$$[\varphi_{T|_V} \leq c] \cap \overline{V} \times X^* \subset T + N_{\overline{V}} \subset D(T) \times X^*.$$

In particular, for every $V \subset S \subset \overline{V}$, S locates T and identifies $T + N_{\overline{V}}$.

Proof. Seeking a contradiction assume that there is $z = (x, x^*) \in [\varphi_{T|_V} < c] \cap \overline{V} \times X^*$. Since T is V -NI we know that $x \in \text{bd } V$.

Let $y \in V \cap D(T)$, $w = (y, y^*) \in T$, and $h : [0, 1] \rightarrow \mathbb{R}$, $h(t) = (\varphi_{T|_V} - c)(tw + (1 - t)z)$. Note that $h(0) < 0$, $h(1) = 0$, since $w \in T|_V \in \mathcal{M}(X)$, and h is continuous. Hence there is $\delta \in (0, 1)$ such that $h(\delta) < 0$. That provides the contradiction $\delta w + (1 - \delta)z \in [\varphi_{T|_V} < c] \cap V \times X^*$.

Note that $h := \varphi_{T|_V} + \iota_{\overline{V} \times X^*} \in \mathcal{R}$, $[h = c] = [\varphi_{T|_V} \leq c] \cap \overline{V} \times X^*$, and

$$h^\square(x, x^*) = \min\{\psi_{T|_V}(x, u^*) + \sigma_{\overline{V}}(x^* - u^*) \mid u^* \in X^*\}, (x, x^*) \in X \times X^*.$$

For every $z = (x, x^*) \in [h^\square = c] \cap \overline{V} \times X^*$ there is $v^* \in X^*$ such that $\psi_{T|_V}(x, v^*) + \sigma_{\overline{V}}(x^* - v^*) = \langle x, x^* \rangle$. This implies that $(x, v^*) \in [\psi_{T|_V} = c] \cap \overline{V} \times X^* \subset [\psi_{T|_{\overline{V}}} = c] \cap \overline{V} \times X^*$ and $x^* - v^* \in N_{\overline{V}}(x)$. If, in addition, T is \overline{V} -representable then $[\psi_{T|_{\overline{V}}} = c] \cap \overline{V} \times X^* = T|_{\overline{V}}$ so $z \in T + N_{\overline{V}}$. Hence

$$[\varphi_{T|_V} \leq c] \cap \overline{V} \times X^* = [h = c] \subset [h^\square = c] \cap \overline{V} \times X^* \subset T + N_{\overline{V}} \subset D(T) \times X^*.$$

■

Proposition 30 *Let X be a LCS, let $T : X \rightrightarrows X^*$, and let $V \subset X$ be open convex such that $D(T) \cap V \neq \emptyset$ and T is \overline{V} -representable. If V locates T then, for every $V \subset S \subset \overline{V}$, S locates T and identifies $T + N_{\overline{V}}$.*

Remark 31 Let $T = (0, 1) \times \{0\} \subset \mathbb{R}^2$. Then $V = (0, 1) = D(T)$ is open convex and identifies T (and, according to Remark 15, that makes T become V -representable) while $\overline{V} = [0, 1]$ does not locate T since $z = (0, 0)$ is m.r. to T and $0 \in \overline{V} \setminus D(T)$.

This example shows the necessity of the \overline{V} -representability condition in the previous two propositions and also, that this condition cannot be replaced by V -representability.

Theorem 32 Let X be a LCS and let $T : X \rightrightarrows X^*$.

- (i) T is locally-NI iff, for every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$, T is C -NI.
- (ii) T is locatable iff, for every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$,

$$[\varphi_{T|_C} \leq c] \cap \text{int } C \times X^* \subset D(T) \times X^*.$$

In particular if, for every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$, C locates T then T is locatable.

- (iii) T is identifiable iff, for every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$,

$$[\varphi_{T|_C} \leq c] \cap \text{int } C \times X^* \subset \text{Graph}(T),$$

In particular if, for every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$, C identifies $T + N_C$ then T is identifiable.

- (iv) T is monotone and identifiable iff T is representable and, for every closed convex $C \subset X$ with $D(T) \cap \text{int } C \neq \emptyset$, C locates T iff T is monotone low-representable and, for every closed convex $C \subset X$ with $D(T) \cap \text{int } C \neq \emptyset$, T is C -NI.

Proof. First we prove that for every open convex $V \subset X$ such that $D(T) \cap V \neq \emptyset$ and for every $x \in V$ there is a closed convex $C \subset V$ such that $D(T) \cap \text{int } C \neq \emptyset$ and $x \in \text{int } C$. Indeed, take $y \in D(T) \cap V$ and a closed convex $U \in \mathcal{V}(0)$ such that $C := [x, y] + U \subset V$. Note that C is closed convex and $y \in D(T) \cap \text{int } C$. The last inclusion is possible since if we assume the opposite, namely that for every $U \in \mathcal{V}(0)$ there is $x_U \in ([x, y] + U) \setminus V$, that is, $x_U - y_U \in U$ for some $y_U \in [x, y]$, because $[x, y]$ is compact, on a subnet, denoted by the same index for notation simplicity, $y_U \rightarrow y \in [x, y] \subset V$ and so we reach the contradiction $x_U \rightarrow y \notin V$.

(i) (\Rightarrow) For every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$, T is $\text{int } C$ -NI. According to Proposition 29, T is also $C = \text{cl}(\text{int } C)$ -NI.

(\Leftarrow) For every $V \subset X$ open convex such that $D(T) \cap V \neq \emptyset$ and every $z = (x, x^*) \in V \times X^*$ let $C \subset V$ be closed convex such that $D(T) \cap \text{int } C \neq \emptyset$ and $x \in C$. Since T is C -NI and $T|_C \subset T|_V$ we get $\varphi_{T|_V}(z) \geq \varphi_{T|_C}(z) \geq c(z)$, i.e., T is locally-NI.

(ii) (\Rightarrow) For every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$, we have $[\varphi_{T|_C} \leq c] \cap \text{int } C \times X^* \subset [\varphi_{T|_{\text{int } C}} \leq c] \cap \text{int } C \times X^* \subset D(T) \times X^*$ since $\text{int } C$ locates T .

(\Leftarrow) For every $V \subset X$ open convex such that $D(T) \cap V \neq \emptyset$ and every $z = (x, x^*) \in [\varphi_{T|_V} \leq c] \cap V \times X^*$ let $C \subset V$ be closed convex such that $D(T) \cap \text{int } C \neq \emptyset$ and $x \in \text{int } C$. Then $z \in [\varphi_{T|_C} \leq c] \cap \text{int } C \times X^* \subset D(T) \times X^*$. This yields that $[\varphi_{T|_V} \leq c] \cap V \times X^* \subset D(T) \times X^*$, that is, V locates T .

The proof of (iii) is similar to the argument used for (ii). In particular if, for every closed convex $C \subset X$ such that $D(T) \cap \text{int } C \neq \emptyset$, C identifies $T + N_C$ then for every $V \subset X$ open convex such that $D(T) \cap V \neq \emptyset$ and every $z = (x, x^*) \in [\varphi_{T|_V} \leq c] \cap V \times X^*$ let $C \subset V$ be closed convex such that $D(T) \cap \text{int } C \neq \emptyset$ and $x \in \text{int } C$. Hence $z \in [\varphi_{T|_C} \leq c] \cap \text{int } C \times X^* = [\varphi_{T+N_C} \leq c] \cap \text{int } C \times X^* \subset \text{Graph}(T)$.

Subpoint (iv) is a direct consequence of (i) and Theorems 23, 25, 27. ■

4 Representability via the convolution operation

The goal of this section is to study, in the context of locally convex spaces, the representability of the sum $A + B$ of two representable operators A, B under classical qualification constraints. Under a Banach space settings, the calculus rules of representable operators can be found in [18, Section 5]. The following result holds

Proposition 33 (*Zălinescu [25, Proposition 1]*) *Let X_1, X_2 be LCS's and let $f_1, f_2 : X = X_1 \times X_2 \rightarrow \overline{\mathbb{R}}$ be proper convex functions. If there exists $(x_1, x_2) \in \text{dom } f_1 \cap \text{dom } f_2$ such that $f_1(\cdot, x_2)$ is continuous at x_1 and $f_2(x_1, \cdot)$ is continuous at x_2 then, for every $x^* \in X^* = X_1^* \times X_2^*$*

$$(f_1 + f_2)^*(x^*) = \min\{f_1^*(u^*) + f_2^*(x^* - u^*) \mid u^* \in X^*\}.$$

Here “min” stands for an infimum that is attained when finite.

Theorem 34 *Let E, F be LCS's and let $\phi_1, \phi_2 : E \times F \rightarrow \overline{\mathbb{R}}$ be proper convex functions. Consider $\rho : E \times F \rightarrow \overline{\mathbb{R}}$ defined by*

$$\rho(x, y) := \inf\{\phi_1(x, y_1) + \phi_2(x, y_2) \mid y_1 + y_2 = y\}.$$

Assume that there exists $(x_0, y_0) \in \text{dom } \phi_2$ such that $x_0 \in \text{Pr}_E(\text{dom } \phi_1)$ and $\phi_2(\cdot, y_0)$ is continuous at x_0 . Then, for every $x^ \in E^*, y^* \in F^*$*

$$\rho^*(x^*, y^*) = \min\{\phi_1^*(x_1^*, y^*) + \phi_2^*(x_2^*, y^*) \mid x_1^* + x_2^* = x^* \in X^*\}.$$

Proof. Note first that $\text{dom } \rho \neq \emptyset$ because there is $\tilde{y} \in F$ such that $(x_0, \tilde{y}) \in \text{dom } \phi_1$, so $\rho(x_0, \tilde{y} + y_0) \leq \phi_1(x_0, \tilde{y}) + \phi_2(x_0, y_0) < +\infty$; whence ρ^* does not take the value $-\infty$.

For every $x \in E, y_1, y_2 \in F, x_1^*, x_2^* \in E^*, y^* \in F^*$ we have

$$\phi_1(x, y_1) + \phi_2(x, y_2) + \phi_1^*(x_1^*, y^*) + \phi_2^*(x_2^*, y^*) \geq \langle x, x_1^* + x_2^* \rangle + \langle y_1 + y_2, y^* \rangle,$$

so, for every $(x, y) \in E \times F, x_1^*, x_2^* \in E^*, y^* \in F^*$,

$$\rho(x, y) + \phi_1^*(x_1^*, y^*) + \phi_2^*(x_2^*, y^*) \geq \langle x, x_1^* + x_2^* \rangle + \langle y, y^* \rangle,$$

from which, for every $x_1^*, x_2^* \in E^*, y^* \in F^*, \phi_1^*(x_1^*, y^*) + \phi_2^*(x_2^*, y^*) \geq \rho^*(x_1^* + x_2^*, y^*)$. Hence

$$\forall (x^*, y^*) \in E^* \times F^*, \quad \rho^*(x^*, y^*) \leq \inf\{\phi_1^*(x_1^*, y^*) + \phi_2^*(x_2^*, y^*) \mid x_1^* + x_2^* = x^* \in X^*\}.$$

If $\rho^*(x^*, y^*) = +\infty$ the conclusion holds.

If $\rho^*(x^*, y^*) \in \mathbb{R}$ consider $f_1, f_2 : (E \times F) \times F \rightarrow \overline{\mathbb{R}}$ given by $f_1(x, y; z) = \phi_2(x, z), f_2(x, y; z) = \phi_1(x, y) - \langle x, x^* \rangle - \langle y + z, y^* \rangle$. Notice that, for every $u^* \in E^*, v^*, z^* \in F^*$

$$\begin{aligned} f_1^*(u^*, v^*; z^*) &= \phi_2(u^*, z^*) + \iota_{\{0\}}(v^*) \\ f_2^*(u^*, v^*; z^*) &= \phi_1(x^* + u^*, y^* + v^*) + \iota_{\{0\}}(y^* + z^*), \end{aligned}$$

$$\begin{aligned} (f_1 + f_2)^*(0) &= -\inf\{\phi_1(x, y) + \phi_2(x, z) - \langle x, x^* \rangle - \langle y + z, y^* \rangle \mid x \in E, y, z \in F\} \\ &= -\inf\{\rho(x, v) - \langle x, x^* \rangle - \langle v, y^* \rangle \mid x \in E, v \in F\} = \rho^*(x^*, y^*) \in \mathbb{R}. \end{aligned}$$

Let $\zeta_1 := (x_0, \tilde{y}) \in \text{dom } \phi_1$ and let $\zeta_2 = y_0$. Then $(\zeta_1, \zeta_2) \in \text{dom } f_1 \cap \text{dom } f_2, \zeta = (x, y) \rightarrow f_1(x, y; \zeta_2) = \phi_2(x, y_0)$ is continuous at ζ_1 , and $f_2(x_0, \tilde{y}; \cdot)$ is continuous at ζ_2 . From Proposition 33 we obtain $u^* \in E^*, v^*, z^* \in F^*$ such that $\rho^*(x^*, y^*) = f_1^*(u^*, v^*; z^*) + f_2^*(-u^*, -v^*; -z^*)$, i.e., $v^* = 0, z^* = y^*$ and $\rho^*(x^*, y^*) = \phi_1(x^* - u^*, y^*) + \phi_2(u^*, y^*)$. ■

Theorem 35 *Let X be a LCS, let $A : X \rightrightarrows X^*$ be representable, and let $C \subset X$ be closed convex. If $D(A) \cap \text{int } C \neq \emptyset$ then $A + N_C$ is representable.*

Proof. Let $x_0 \in D(A) \cap \text{int } C$, $a_0^* \in Ax_0$. We apply the previous theorem for $E = X$, $F = (X^*, w^*)$, $\phi_1 = \varphi_A$, $\phi_2(x, x^*) = \iota_C(x) + \sigma_C(x^*)$, $(x, x^*) \in X \times X^*$, and $y_0 = 0 \in \text{dom } \sigma_C$ to get that

$$\rho(x, x^*) := \inf\{\varphi_A(x, x_1^*) + \iota_C(x) + \sigma_C(x_2^*) \mid x_1^* + x_2^* = x^*, (x, x^*) \in X \times X^*,$$

has

$$\rho^\square(x, x^*) := \min\{\psi_A(x, x_1^*) + \iota_C(x) + \sigma_C(x_2^*) \mid x_1^* + x_2^* = x^*, (x, x^*) \in X \times X^*,$$

which is a representative of $A + N_C$. ■

Theorem 36 *Let X be a barreled LCS and let $A, B : X \rightrightarrows X^*$ be representable such that $D(A) \cap \text{int } D(B) \neq \emptyset$. For every $x \in X$, there exist $V \in \mathcal{V}(x)$, such that $A + B$ is V -representable. In particular, $A + B$ is low-representable.*

Proof. Fix

$$x_0 \in D(A) \cap \text{int } D(B), a_0^* \in Ax_0, b_0^* \in Bx_0, x_0^* = a_0^* + b_0^*, z_0 := (x_0, x_0^*) \in A + B,$$

and symmetric open convex $U_0, U_1 \in \mathcal{V}(0)$ such that $U_1 + U_1 \subset U_0$, $x_0 + U_0 \subset D(B)$, and $B(x_0 + U_0)$ is equicontinuous; for simplicity $B(x_0 + U_0) \subset U_0^\circ$.

Take an arbitrary $x \in X$ and denote by $V = [x_0, x] + U_1 \in \mathcal{V}(x)$.

Notice that $(x_0, a_0^*) \in A|_V$ so $x_0 \in \text{Pr}_X(\text{dom } \varphi_{A|_V})$ and $\varphi_{B|_V}(\cdot, 0)$ is continuous at x_0 since $\varphi_{B|_V}(\cdot, 0)$ is bounded from above on $x_0 + U_1$. Indeed, for every $y \in x_0 + U_1 \subset D(B)$, $y^* \in B(y)$, $b \in V \cap D(B)$, $b^* \in B(b)$, we have $y - b \in [0, x_0 - x] + U_0$ and $\langle y - b, b^* \rangle \leq \langle y - b, y^* \rangle$. Hence, for every $y \in x_0 + U_1$, $y^* \in B(y)$

$$\begin{aligned} \varphi_{B|_V}(y, 0) &= \sup\{\langle y - b, b^* \rangle \mid b \in V \cap D(B), b^* \in B(b)\} \\ &\leq \sup\{\langle y - b, y^* \rangle \mid b \in V \cap D(B), b^* \in B(b)\} \\ &\leq \sup\{|\langle x_0 - x, u^* \rangle| + 1 \mid u^* \in B(x_0 + U_0)\} < +\infty. \end{aligned}$$

Consider $\rho : X \times X^* \rightarrow \overline{\mathbb{R}}$,

$$\rho(y, y^*) = \inf\{\varphi_{A|_V}(y, u^*) + \varphi_{B|_V}(y, v^*) \mid u^* + v^* = y^*\}.$$

We apply Proposition 34 for $E = X$, $F = (X^*, w^*)$ to get

$$\rho^\square(y, y^*) = \min\{\psi_{A|_V}(y, u^*) + \psi_{B|_V}(y, v^*) \mid u^* + v^* = y^*\}.$$

Note that $\varphi_{A+B|_V} \leq \rho$, so, $\rho^\square \leq \psi_{A+B|_V}$. Therefore, for every $w = (y, y^*) \in [\psi_{A+B|_V} = c] \cap V \times X^*$ there exists $u^* \in X^*$ such that $(y, u^*) \in [\psi_{A|_V} = c]$, $(y, x^* - u^*) \in [\psi_{B|_V} = c]$, since $\rho^\square \geq c$. Because $y \in V$ and A, B are V -representable, we get that $w \in A + B$, that is, $A + B$ is V -representable. ■

5 Open problems

1. If T is maximal monotone, how can we better describe condition (1)? Is it the same as T is locally-NI?
2. For a fixed open convex V , clearly V locates T implies that T is V -NI which in turns yields condition (1). Are any of the converses of these two implications true?
3. If T is monotone then $(1) \Rightarrow \overline{D(T)}$ is convex?
4. If T is monotone, V is open convex, $V \cap D(T) \neq \emptyset$, and \overline{V} locates T then must V locate T ?
5. Under the hypotheses of Theorem 27, is $T + N_C$ maximal monotone?

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